Mathematics Methods
Unit 3 & 4

Differentiation

1.	Types of differentiation
	(a) First principle (Method 1)
	$y = f(x)$ $y + \delta y = f(x + \delta x)$ Tips: 1. Add, + δy and $+ \delta x$ 2. Make $\frac{\delta y}{\delta x}$ as subject 3. Solve *lim* $\delta x \to 0$ $\frac{\delta y}{\delta x}$ 4. $\frac{dy}{dx} = ans$ from step 3
	Example 1: Differentiate $y = 3x - 6$ using the first principle. Example 2: Find the differentiation of $y = x^2 + 3$.

(b) First principle (Method 2)

Proving that $\frac{d}{dx} \sin x = \cos x$

$$f'(x) = \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h}$$
$$= \lim_{h \to 0} \frac{\sin x \cos h + \sin h \cos x - \sin x}{h}$$
$$= \lim_{h \to 0} \frac{\sin x (\cos h - 1) + \sin h \cos x}{h}$$
$$= \lim_{h \to 0} \frac{\sin x}{1} \frac{(\cos h - 1)}{h} + \lim_{h \to 0} \frac{\sin h}{h} \frac{(\cos x)}{1}$$
$$= \sin x \, [\lim_{h \to 0} \frac{\cos h - 1}{h}] + \cos x \, [\lim_{h \to 0} \frac{\sin h}{h}]$$
$$= \sin x \,(0) + \cos x \,(1)$$
$$= \cos x$$

Proving that $\frac{d}{dx} \cos x = -\sin x$

$$f'(x) = \lim_{h \to 0} \frac{\cos(x+h) - \cos(x)}{h}$$
$$= \lim_{h \to 0} \frac{\cos x \cos h - \sin x \sin h - \cos x}{h}$$
$$= \lim_{h \to 0} \frac{\cos x \cos h - \cos x - \sin x \sin h}{h}$$
$$= \lim_{h \to 0} \frac{\cos x (\cos h - 1) - \sin x \sin h}{h}$$
$$= \lim_{h \to 0} \cos x \, [\frac{(\cos h - 1)}{h}] - \lim_{h \to 0} \sin x \, [\frac{\sin h}{h}]$$
$$= \lim_{h \to 0} \cos x \frac{(\cos h - 1)}{h} - \lim_{h \to 0} \sin x \frac{\sin h}{h}$$
$$= \cos x \,(0) - \sin x \,(1)$$
$$= -\sin x \,(1)$$

Proving that $\frac{d}{dx} \tan x = \sec^2 x$

$$f'(x) = \lim_{h \to 0} \frac{\tan(x+h) - \tan(x)}{h}$$
$$= \lim_{h \to 0} \frac{\frac{\tan x + \tan h}{1 - \tan x \tan h} - \tan x}{h}$$
$$= \lim_{h \to 0} \frac{\frac{\tan x + \tan h}{1 - \tan x \tan h} - \tan x \frac{1 - \tan x \tan h}{1 - \tan x \tan h}}{h}$$

$$= \lim_{h\to 0} \frac{tan\,x + tan\,h - tan\,x \;+\; tan^2x\,tan\,h}{h(1-\;tan\,x\,tan\,h)}$$

$$= \lim_{h\to 0} \frac{tan\,h\,(1+\;tan^2x\,)}{h(1-\;tan\,x\,tan\,h)}$$

$$= \lim_{h\to 0} \frac{sin\,h\,(1+\;tan^2x\,)}{cos\,h\,h(1-\;tan\,x\,tan\,h)}$$

$$= \lim_{h\to 0} \frac{sin\,h\,(1+\;tan^2x\,)}{cos\,h\,h(1-\;tan\,x\,tan\,h)}$$

$$= \lim_{h\to 0} \frac{sin\,h}{h} \lim_{h\to 0} \frac{1+\;tan^2x}{cos\,h\,(1-\;tan\,x\,tan\,h)}$$

$$= 1 \times \frac{1+\;tan^2x}{cos\,0\,(1-\;tan\,x\,tan\,0)}$$

$sec^2x = 1 + \;tan^2x,$

$$= sec^2x$$

Proving that $\frac{d}{dx} e^x = e^x$

$$f'(x) = \lim_{h\to 0} \frac{e^{x+h} - e^x}{h}$$
$$= \lim_{h\to 0} e^x \left(\frac{e^h - 1}{h}\right)$$
$$= e^x(1)$$
$$= e^x$$

Proving that $\frac{d}{dx} \ln x = \frac{1}{x}$

$$f'(x) = \lim_{h\to 0} \frac{\ln\,(x+h) - \ln x}{h}$$

$$= \lim_{h\to 0} \frac{\ln\,(\frac{x+h}{x})}{h}$$

$$= \lim_{h\to 0} \frac{\ln\,(1+\frac{h}{x})}{h}$$

$$= \lim_{h\to 0} \frac{\ln\,(1+\frac{h}{x})}{\frac{h}{x}} \frac{1}{x}$$

$$= (1)\;\frac{1}{x}$$

$$= \frac{1}{x}$$

Example:
Differentiate $y = x^2 + 2x$ using first principle method.

(c) Differentiation of logarithm, trigonometric functions and exponential functions.

Trigonometric functions

$$\frac{d}{dx} sin\ x = cos\ x$$

$$\frac{d}{dx} cos\ x = -sin\ x$$

$$\frac{d}{dx} tan\ x = sec^2\ x$$

$$\frac{d}{dx} sec\ x = sec\ x\ tan\ x$$

Exponential function

$$\frac{d}{dx} e^x = e^x$$

$$\frac{d}{dx} e^{ax} = ae^{ax}$$

$$\frac{d}{dx} e^{f(x)} = f'(x) \cdot e^{f(x)}$$

Logarithmic function

$$\frac{d}{dx} \ln x = \frac{1}{x}$$

(d) Using formula

(i) "Formula 1"

$$y = ax^n + b$$
$$\frac{dy}{dx} = anx^{n-1}$$

Example 1:
Differentiate $y = 7x^5$.

Example 2:
Differentiate $y = 5x^{-4} + 19$.

Example 3:
Differentiate $y = 5x^2 + 7x + 36$.

(ii) "Formula 2"

$$y = a(bx + c)^n$$
$$\frac{dy}{dx} = an(bx + c)^{n-1}(b)$$
$$= nab\,(bx + c)^{n-1}$$

Example 1:
Differentiate $y = 4(3x + 9)^7$.

Example 2:
Differentiate $y = (3x^3 + 4x + 3)^{-4}$.

(e) Product rule

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

Basic polynomial functions

Example 1:
Differentiate $y = x^2(2x - 3)$ using product rule.

Example 2:
Differentiate $y = (2x + 7)(7 - x)^3$.

Trigonometric functions

Form of $a\ (sin/cos/tan)\ x$	Form of $a^n\ (sin/cos/tan)\ x$
Example 1: Differentiate $y = 15\ sin\ x$. Example 2: Differentiate $y = 2x\ tan\ x$.	Example: Differentiate $y = x^3\ cos\ x$.

Exponential functions

Form of $f(x) \cdot e^x$	
Example 1: Differentiate $y = \sqrt[3]{x}\, e^x$.	Example 2: Differentiate $y = cos^2 x\, e^x$.
Example 3: Differentiate $y = 4x^2\, e^x$.	

Logarithmic functions

Form of $a \; \ln x$	
Example 1: Differentiate $y = (x+1)\; \ln x$.	Example 2: Differentiate $y = 2e^{\frac{x}{3}}\; \ln x$.
Example 3: Differentiate $y = 3 \ln x$.	

(f) Division/ quotient rule

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Example 1:
Differentiate $y = \frac{4x}{2x-1}$ using division rule.

Example 2:
Differentiate $y = \frac{3x+6}{6x-87}$.

(g) Chain rule

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

Example:
Differentiate $(2x+2)^4$.

Trigonometric functions

Form of $(sin/cos/tan)ax$	**Form of $sin^n/\ cos^n/\ tan^n\ x$**
Example 1: Differentiate $y = sin\ 3x$.	Example 1: Differentiate $y = tan^3 x$.
Example 2: Differentiate $y = cos(3x - 4)$.	Example 2: Differentiate $y = sin^2\ x$.

Exponential function

Form of ae^{bx}	**Form of $ae^{f(x)}$**
Example 1: Differentiate $y = 5e^{-8x}$.	Example 1: Differentiate $y = e^{cosx}$.
Example 2: Differentiate $y = e^{4x}$.	Example 2: Differentiate $y = e^{x^3 + sin\ x}$.

Logarithmic function

Form of $\ln(ax+b)$	
Example 1: Differentiate $y = \ln(e^{2x} + 2)$.	Example 2: Differentiate $y = \ln(x + 2x^2)$.

2. Gradient

(a) Determine gradient function of any equation

Gradient function be m,

$$m = \frac{dy}{dx}$$

Example:
Determine the gradient function of the following equations:

(a) $y = (x^5 + 3)(x + 3)$

(b) $y = \frac{19}{t^2}$

(c) $y = e^{3u} + 2u^3$

(d) $y = e^x + 2x$

(e) $y = 2^x$

(f) $y = 2\ln u$

(b) Determine gradient of an equation at a given point

Tips:

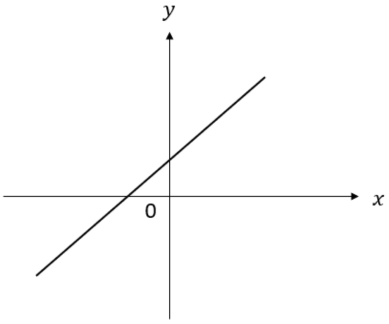
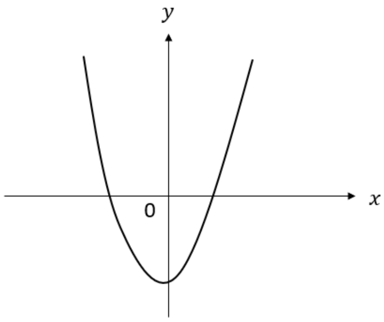
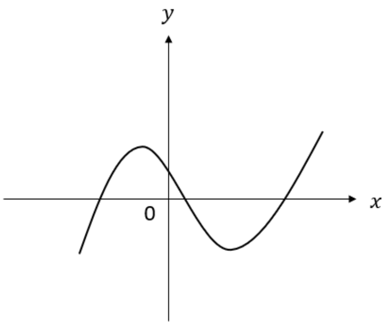
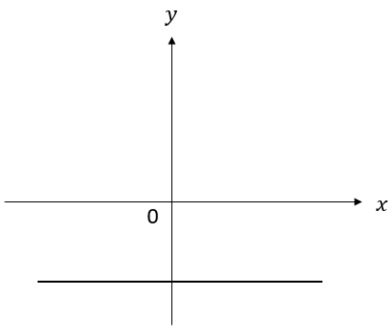
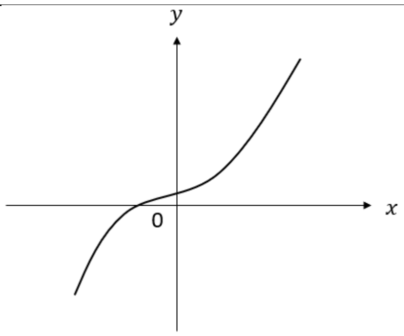
1. Differentiate, $\frac{dy}{dx}$ the equation of equation.
2. Substitute the x value of a given point (x, y) into $\frac{dy}{dx}$

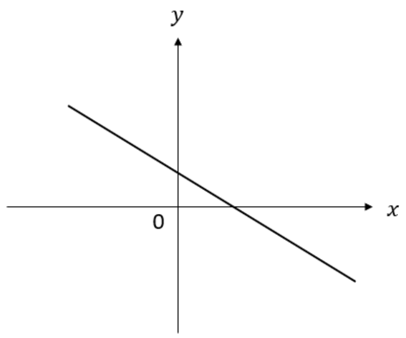
Example 1:
Determine the gradient of the curve $y = 6x^2 + 3x - 2$ at $(9, 3)$.

Example 2:
Determine the gradient of $y = \frac{4x+2}{x}$ at $(2, 5)$.

Example 3:
Determine the gradient of curve, $y = e^x + 2x$ at $(1,0)$.

(c) Characteristics of the gradient of graphs

Graphs	Characteristics of gradient
	• Always positive • Never negative • Independent of x
	• Gradient is zero, 0 at stationary point
	• Gradient is zero, 0 at maximum and minimum point
	• Independent of x • Never negative
	• Always positive • Never negative

	• Always negative • Never positive • Independent of x

3. First and second derivative

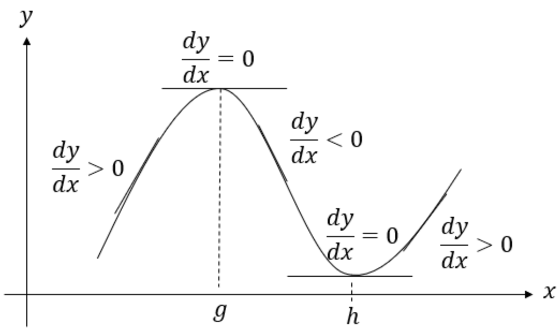
<u>Definition of first derivative and second derivative</u>

$\frac{dy}{dx}$ depicts how y changes when x is increased by 1 unit

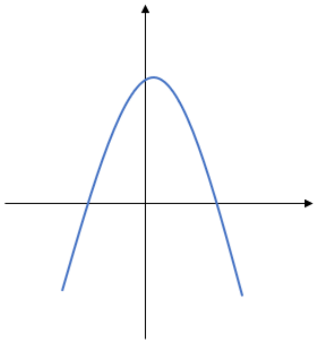
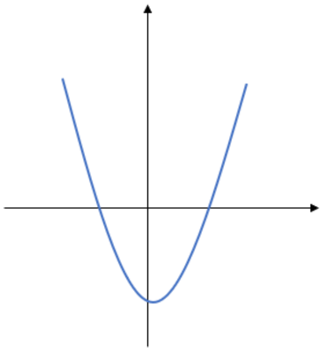
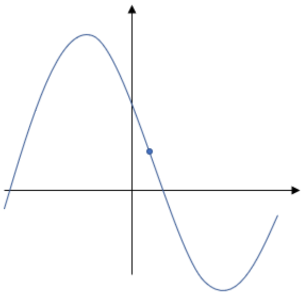
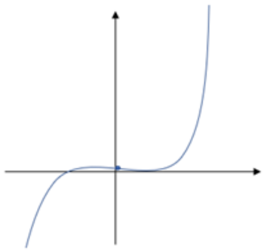
$\frac{d^2y}{dx^2}$ depicts how $\frac{dy}{dx}$ changes when x is increased by 1 unit

$\frac{d^2y}{dx^2}$ is known as "gradient function of the gradient function" when the equation of gradient $\frac{dy}{dx}$ is differentiated again.

First and second derivative on graph



$\frac{dy}{dx} = 0$ means gradient is zero at points g (max. point) and h (min. point).
$\frac{dy}{dx} > 0$ means gradient is positive
$\frac{dy}{dx} < 0$ means gradient is negative

First derivative	Represents
$f'(x) = 0 \quad / \quad \frac{dy}{dx} = 0$	Stationary points (can be max. point, min. point and inflection point)
Second derivative	**Represents**
$f''(x) < 0 \quad / \quad \frac{d^2y}{dx^2} < 0$	• Maximum point • Graph concaves down 
$f''(x) > 0 \quad / \quad \frac{d^2y}{dx^2} > 0$	• Minimum point • Graph concaves up 
$f''(x) = 0 \quad / \quad \frac{d^2y}{dx^2} = 0$	Inflection point (either horizontal inflection point or oblique inflection point)  or 

(a) Stationary points

$$f'(x) = 0 \quad / \quad \frac{dy}{dx} = 0$$

Example 1:
Determine the coordinates of stationary points for the curve $y = x^3 - 12x^2 + 36x - 15$.

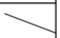
Example 2:
Determine the coordinates of stationary points for the curve $y = 2x\,e^x$.

(b) Maximum or minimum points

There are two ways to determine whether the stationary points are max. or min.:

<table>
<tr><th>Sign test using first derivative</th><th>Nature of stationary point</th><th>Second derivative test</th></tr>
<tr><td>

$$\frac{dy}{dx} = 0$$

<table>
<tr><td></td><td>x^-</td><td>x_0</td><td>x^+</td></tr>
<tr><td>$\frac{dy}{dx}$</td><td>$+ve$</td><td>0</td><td>$-ve$</td></tr>
<tr><td>Slope</td><td></td><td></td><td></td></tr>
<tr><td>Stationary point</td><td colspan="3"></td></tr>
</table>

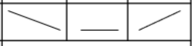
</td><td>Maximum point</td><td>

$$\frac{d^2y}{dx^2} < 0$$

</td></tr>
</table>

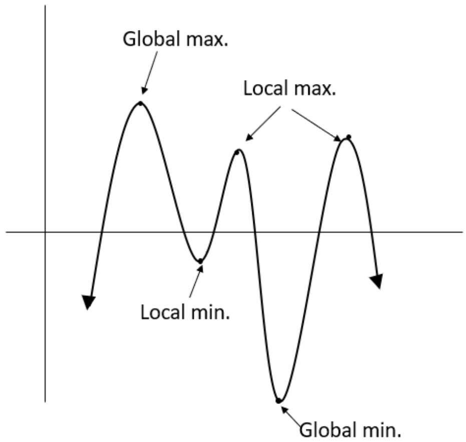
<table>
<tr>
<td>

$$\frac{dy}{dx} = 0$$

<table>
<tr><td></td><td>x^-</td><td>x_0</td><td>x^+</td></tr>
<tr><td>$\frac{dy}{dx}$</td><td>$-ve$</td><td>0</td><td>$+ve$</td></tr>
<tr><td>Slope</td><td colspan="3"></td></tr>
<tr><td>Stationary point</td><td colspan="3"></td></tr>
</table>

</td>
<td>Minimum point</td>
<td>$\frac{d^2y}{dx^2} > 0$</td>
</tr>
</table>

<u>Characteristics</u>



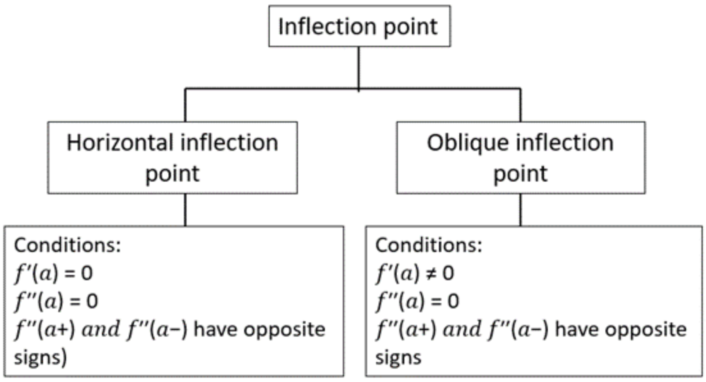
Local	Maximum	The point is largest of the function within a specific range
	Minimum	The point is smallest of the function within a specific range
Global	Maximum	The largest point of function on the entire domain
	Minimum	The smallest point of function on the entire domain

Example:

Determine the nature of stationary points of the curve $f(x) = x + \frac{5}{x}$.

(c) Inflection point

$$f''{}'(x) = 0 \quad / \quad \frac{d^2y}{dx^2} = 0$$



Steps to find inflection point:

1. Find $f''(x)$ / $\frac{d^2y}{dx^2}$.
2. Make $f''(x) = 0$ / $\frac{d^2y}{dx^2} = 0$ to find the value of x.
3. Substitute x into $f(x)$. A coordinate $\big(x, f(x)\big)$ is found.
4. Substitute x from $\big(x, f(x)\big)$ into $f'(x)$.
5. Use value x from $\big(x, f(x)\big)$ to form a table:

x	$x - 0.01$	x	$x + 0.01$
$f''(x)$ (+ve or -ve)			

6. Determine whether the conditions for horizontal/ oblique inflection are met:

Horizontal inflection point	Oblique inflection point
$f'(a) = 0$ $f''(a) = 0$ $f''(a+)$ *and* $f''(a-)$ have opposite signs	$f'(a) \neq 0$ $f''(a) = 0$ $f''(a+)$ *and* $f''(a-)$ have opposite signs

Example 1:

Find the point inflection for $f(x) = \frac{1}{2}x^4 + x^3 - 6x^2$.

4.	**Sketching a graph**
	(a) Given an equation of graph
	Steps: 1. Determine the coordinates of the y -axis intercept and $x -$ axis intercept. 2. Determine the behaviour of the function as $x \rightarrow \pm\infty$ (sub 9.99999 into function/ equation). 3. Determine the location and nature of any turning points. 4. Determine the coordinates of any points for which $\frac{d^2y}{dx^2} = 0$.
	Example: Plot the graph of $y = x(x-3)^2$.

(b) Given certain characteristics of graph

Example:
Sketch a graph given the conditions:

- $f'(x) > 0$ for $-3 < x < 3$ and $x > 7$
- $f(-3) = f(6) = f(8) = 0$
- $f(0) = 3$
- $f'(3) = f'(7) = 0$
- $f''(x) < 0$ for $x = 3$ and $f''(x) > 0$ for $x = 7$

5. Small change, rate of change

Small change,

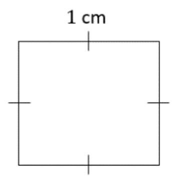
$$\frac{\delta y}{\delta x} \approx \frac{dy}{dx}$$

Approximation,

$$Y_{new} = Y_{old} + \delta y$$

(a) Approximate change

Example:
Find the approximate change in the area of square when the sides are increased from 1 cm to 1.1 cm.



(b) Approximate percentage change

Example:
Determine the approximate percentage change of $y = 3x^2$ when x increases by 1%.

(c) Rate of change

Example 1:
Given that $f'(x) = x^3 + 3$, find the rate of change of $f'(x)$ when $x = 3$.

Example 2:
A drop of methyl blue solution is dripped into a bowl of water and small circle ripples are formed continuously. Given that the radius of ripple increases at rate of 0.5 cms^{-1}, find the rate of change of the area of ripple in terms of π when the diameter is 0.4 cm

6. Economic applications of differentiation

Summary:

Application	**Function**
Production cost (cost of producing x unit of items)	$C(x)$
Revenue (earnings from the sales of x unit of items)	$R(x)$
Profit	$P(x) = C(x) - R(x)$

Break even (neither profit or loss is incurred, cost of producing x unit of items equals to earnings from the sales of x unit of items)	$C(x) = R(x)$
Average cost (cost of producing x unit of items)	$\frac{c(x)}{x}$
Marginal Profit (The rate of change of profit with respect to the number of units sold)	$P'(x)$
Marginal Revenue (the approximate additional revenue brought in by the sale of one more item after the x th item has been sold)	$R'(x)$
Marginal cost (rate of change of total cost with respect to the total units produced)	$C'(x)$
Approximate cost of producing one more unit after x th unit has been produced and sold	$C''(x)$
Maximum profit	$C'(x) = R'(x)$ *or* $P'(x)$

(a) Cost

(i) Production cost, $C(x)$

Definition: cost of producing x unit of items

Production cost function:

$$C(x) = variable\ cost\ +\ fixed\ cost$$

Example:
A.M Ltd. manufactures stylish notebooks. The company's total fixed cost is $5,000 and the variable cost of each notebook is $1.5. Equate the cost function for the production of the notebooks.

(ii) Average cost, $\frac{c(x)}{x}$

Definition: cost of producing x unit of items

		Example: The cost of producing an item is given by $C(x) = 2x^2 + 5x - 3$. Find the average cost of producing 500 units of the item.
	(iii)	**Marginal cost, $C'(x)$** Definition: rate of change of total cost with respect to the total units produced
		Example 1: Cost of producing a stock of products can be equated by $C(x) = 5x^2 + 5$. Find the marginal cost function. Example 2: The cost of producing x units of products can be given by $C(x) = 10x^3 - 50x^2 + 62x$. Find the value of x that gives a maximum marginal cost.
	(b) Revenue	
	(i)	**Revenue, $R(x)$** Definition: earnings from the sales of x unit of items Revenue function: $$R(x) = kx$$ k: selling price
		Example 1: Sally plans to sell lemonades at $\$(3.5 - 0.2x)$ per cup. Determine the revenue function.

Example 2:
Adlan Wace sells digital online notes at $30 per subject. Determine the revenue function.

(ii) Marginal Revenue, $R'(x)$

Definition: the approximate additional revenue brought in by the sale of one more item after the x th item has been sold

Example 1:
The revenue function of x units of baking soda is given by $R(x) = 2x^2 + 10x - 5$. How much does the revenue increases due to the selling of 201th baking soda?

Example 2:
The Buttocion Travels offers sightseeing tours of Perth. One of the tour costs $5 per person and after calculation has an average demand of about 3000 customers per day. The proprietor experimented the demand of customers by lowering the price to $4 the daily demand rose to 5000 customers. Find the tour cost to be charged per customer to maximise the total revenue each day. Assume that the equation of demand is a linear equation.

(c) Profit

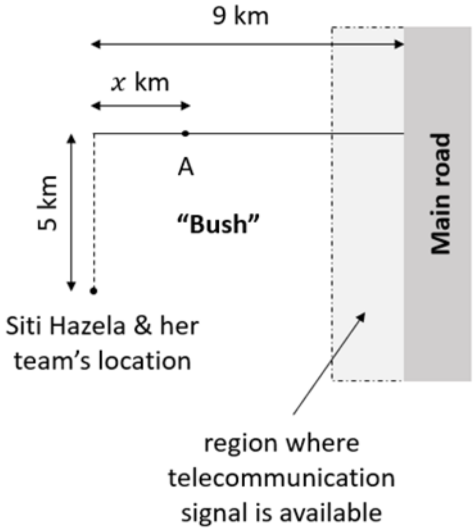
(i) Profit, $P(x)$

Definition: income gained from the sales of x unit of items

Profit function:

$$P(x) = C(x) - R(x)$$

		Example: The cost of making a product is $\$(5x^2 + 3x)$ and each product is sold at \$19. Find the profit function.
	(ii)	**Marginal profit**, $\boldsymbol{P'(x)}$ Definition: the rate of change of profit with respect to the number of units sold
		Example 1: The profit function of selling x units of items is given by $P(x) = 2x^3 + 3x - 2$. Find the marginal profit if 120 units are sold. Example 2: The information below shows two functions: $R(x) = 7000x - 5x^2$ $C(x) = 50x - 20$ Find the number of units, x to be sold that maximizes the marginal profit.

7.	**Optimisation problems**
	Example: Siti Hazela got her leg bitten during the filming of a wildlife documentary. Her location is at the "bush" region where every else other than the "bush" region is the desert region. One of her team members would have to leave her and ride their only 4WD vehicle to the main road to call for help as there is no telecommunication signal at the location except region around the main road and the town. There is only telecommunication signal at maximum the region of 3km near the main road. On the desert, he can only drive at speed of $12kmh^{-1}$ while on the bush, he can drive at speed of $4.8kmh^{-1}$. He can either drive directly to the main road or ride to point A on the dry land then drive straight horizontally to the main road.  Which is the pathway in which shortest time is consumed? Hence, find the shortest time in hours.

END